J Math Chem (2014) 52:1841–1847
DOI 10.1007/s10910-014-0350-0

ORIGINAL PAPER

The spectrum and spanning trees of polyominos on the torus

Fuliang Lu · Yajun Gong · Houchun Zhou

Received: 12 March 2014 / Accepted: 17 March 2014 / Published online: 23 March 2014

Abstract Polyominos was extensively studied in chemistry and mathematics. The spectrum of a (molecule) graph is the set of eigenvalues of its adjacency matrix. The spectrum and the number of spanning trees of polyominos on the torus are determined in this paper.

Keywords Polyominos · Spectra · Spanning trees

1 Introduction

A polyomino, also called quadrilateral lattice, chessboard [3], square-cell configuration or lattice animal [7,8,11], is a finite 2-connected geometric graph in which every interior face is bounded by a regular square of side length 1 (i.e. called a cell). Polyominos have attracted many mathematicians', physicists' and chemists' considerable attentions in history. Many interesting combinatorial subjects are yielded from them, such as domination problem [3,5], spanning trees [12] and rook polyominal [10] etc. Zhang and Zhang [13] gave a necessary and sufficient conditions for polyomino graphs to have a Kekulé structure. Calkin and Wilf [1] counted the number of independent sets in a polyomino graph. Merino and Welsh [9] considered the forest, colourings and acyclic orientations on the polyomino.

An $m \times n$ polyomino, denoted by $P_{m,n}$, consists of mn sites arranged in an array of M rows and N columns, see Fig. 1a. By adding edges a_1a_n, b_1b_n, $u_jv_j (j = 1, 2, \ldots, m-2)$, an $m \times n$ polyomino with cylindrical boundary condition can be gotten, denoted by $P^c_{m,n}$. By adding edges a_1a_n, b_1b_n, $u_jv_j (j = 1, 2, \ldots, m-2)$ and $a_kb_{k+r} (i = 1, 2, \ldots, n, 0 \le r < n)$ to $P_{m,n}$, an $m \times n$ polyomino with the twisted toroidal boundary condition can be gotten, denoted by $P_{m,n,r}$.

F. Lu (✉) · Y. Gong · H. Zhou
School of Sciences, Linyi University, Linyi 276005, Shandong, People's Republic of China
e-mail: flianglu@163.com

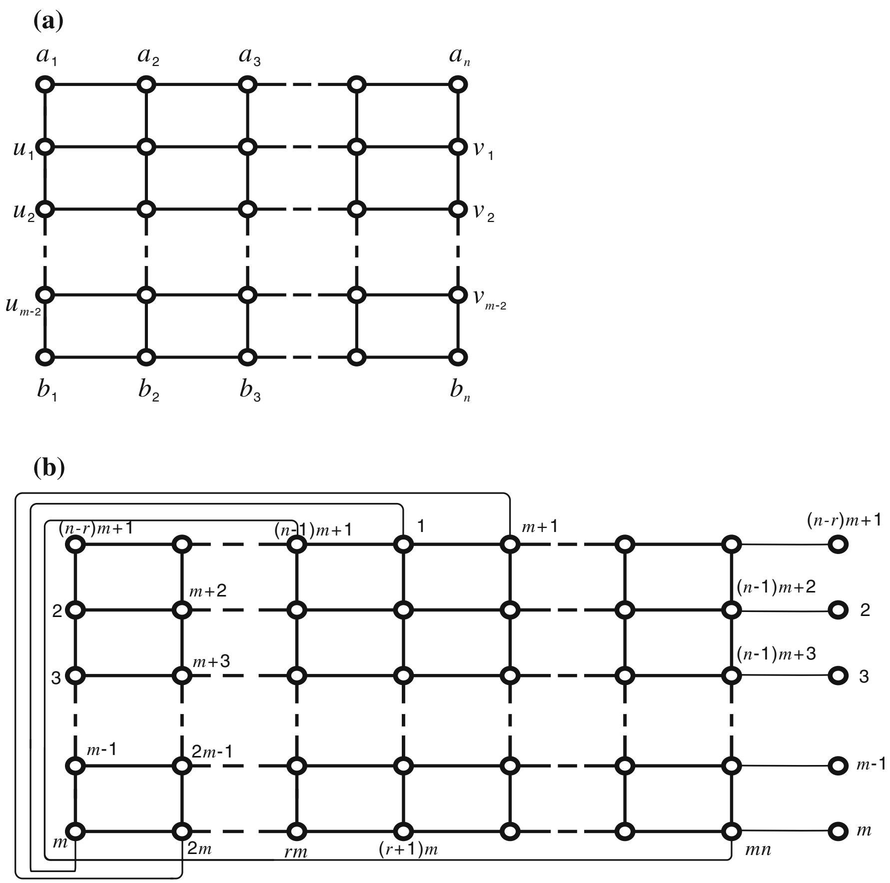

Fig. 1 **a** A $m \times n$ polyomino $P_{m,n}$; **b** the labeling of $P_{m,n,r}$

Let $G = (V(G), E(G))$ denote a (molecule) graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The degree k_s of a vertex v_s is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. The adjacency matrix $A(G)$ of G is the $n \times n$ matrix with elements $A(G)_{sj} = 1$ if v_s and v_j are connected by an edge and zero otherwise. The Laplacian matrix $L(G)$ is the $n \times n$ matrix with the element $L(G)_{sj} = k_s \delta_{sj} - A(G)_{sj}$, where δ_{sj} is the Kronecker delta, equal to 1 if $s = j$ and zero otherwise. Suppose $\lambda_j (j = 1, 2, \ldots, n)$ are the eigenvalues of the adjacency matrix $A(G)$. Suppose $\mu_j (j = 1, 2, \ldots, n)$ are the eigenvalues of the Laplacian matrix $L(G)$, where $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. We know that $\mu_1 = 0$, then the number of spanning trees of a graph G can be expressed by

$$t(G) = \frac{1}{n} \prod_{j=2}^{n} \mu_j.$$

It turns out that $t(G)$ has asymptotically exponential growth; one defines the quantity $z(G)$ by

$$z(G) = \lim_{|V(G)|\to\infty} \frac{\log t(G)}{|V(G)|}.$$

This limit is known as the asymptotic tree number entropy, asymptotic growth constant or thermodynamical limit.

The spectrum of $P^c_{m,n}$ has been gotten (see for example Section 2.6 in [4]). It can be expressed as follows:

$$2\cos\frac{2k\pi}{n} + 2\cos\frac{j\pi}{m+1}, \quad k = 1, \ldots, n, \ j = 1, \ldots, m.$$

And the eigenvalue of $P_{m,n,0}$ can be expressed as follows (see for example Section 2.6 in [4]):

$$2\cos\frac{2k\pi}{n} \pm 2\cos\frac{2j\pi}{m}, \quad k = 0, 1, \ldots, n-1, \ j = 0, 1, \ldots, \frac{m}{2} - 1. \qquad (1)$$

Wu [12] has obtained closed-form expressions for the number of spanning tree of $P_{m,n}$ and the asymptotic tree number entropy is 1.1662.

In this paper, we consider the spectrum and spanning trees of $P_{m,n,r}$. In Sect. 2, we present a lemma. The spectrum of $P_{m,n,r}$ is obtained in Sect. 3 and the number of spanning trees of $P_{m,n,r}$ is gotten in Sect. 4.

2 A lemma

Firstly, we need a lemma. Denote the k block circulant matrix

$$\begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-1} \\ kV_{n-1} & V_0 & V_1 & \cdots & V_{n-2} \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & V_{n-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ kV_1 & kV_2 & \cdots & kV_{n-1} & V_0 \end{pmatrix}$$

by $k - circ(V_0, V_1, \ldots, V_{n-1})$.

Lemma 1 ([2]) *Let $V = k - circ(V_0, V_1, \ldots, V_{n-1})$ be a k block circulant matrix over the complex number field, where all V_t are $m \times m$ matrices, $t = 0, 1, \ldots, n-1$. Then*

$$\det V = \prod_{t=0}^{n-1} \det(J_t),$$

where $J_t = V_0 + V_1\omega + V_2\omega^2 + \cdots + V_{n-1}\omega^{n-1}$, ω is a primitive nth root of k.

3 The spectrum of square lattice

Theorem 2 *If m is even, then the spectrum of the $P_{m,n,r}$ can be expressed by*

$$2\cos\frac{2k\pi}{n} \pm 2\cos\left(\frac{2j\pi}{m}+\frac{2rk\pi}{mn}\right), \quad k=0,1,\ldots,n-1,\ j=0,1,\ldots,\frac{m}{2}-1.$$

Proof Label the vertices of the $P_{m,n,r}$ as Fig. 1b shows. Then the adjacent matrix of it has the following form:

$$A(G) = 1-circ\left(A_m, E_m, \overbrace{0_m,\ldots,0_m}^{r-2}, B_m, 0_m,\ldots,0_m, B_m^T, \overbrace{0_m,\ldots,0_m}^{r-2}, E_m\right),$$

where

$$A_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{m\times m}, \quad B_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times m}.$$

B^T is the transpose of B, E_m is the $m\times m$ identity matrix, 0_m is an $m\times m$ matrix and all its entries are zero.

Next, we calculate the eigenvalue value of $A(G)$. The characteristic polynomial of $A(G)$ is

$$\phi(\lambda) = |\lambda E_{mn} - A(G)|.$$

By Lemma 1, we have

$$\phi(\lambda) = \prod_{k=0}^{n-1} |\lambda E_m - f(\omega_k)|,$$

where $f(x) = A + xE + x^r B + x^{n-r+1}B^T + x^{n-1}E$ and $\omega_k = \cos(2k\pi/n) + i\sin(2k\pi/n)$.

Note that

$$\lambda E_m - f(\omega_k) = \begin{pmatrix} -a & -b & 0 & \cdots & 0 & 0 & -1 \\ -b^{-1} & -a & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -a & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & -a & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & -a & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & -a \end{pmatrix}_{m\times m},$$

where $a = (\omega_k + \omega_k^{n-1}) - \lambda$, $b = \omega_k^r$. Multiplying by $b^{j/2-1}$ in columns j and $j+1$ when j is even and $m \geq j > 2$, then multiplying by b in rows 2 and by $b^{-(j-1)/2+1}$ in rows j and $j+1$ when j is odd and $m > j > 3$, we have

$$b^{\frac{m}{2}}|\lambda E_m - f(\omega_k)| = \begin{vmatrix} -a & -b & 0 & \cdots & 0 & 0 & 0 & 0 & -b^{\frac{m}{2}-1} \\ -1 & -ab & -b & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & -a & -b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & -ab & -b & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & -a & -b & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & -ab & -b & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -a & -b \\ -b^{2-\frac{m}{2}} & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -ab \end{vmatrix}_{m\times m}$$

$$= \left| b^{1-\frac{m}{2}} - circ(A', B', 0, \ldots, 0, C') \right|,$$

where

$$A' = \begin{pmatrix} -a & -b \\ -1 & -ab \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & -b^{\frac{m}{2}-1} \\ 0 & 0 \end{pmatrix}.$$

By Lemma 1,

$$b^{\frac{m}{2}}|\lambda E_m - f(\omega_k)| = b^{\frac{m}{2}} \prod_{j=0}^{\frac{m}{2}-1} \left| A' + B'\varepsilon_j + C'\varepsilon_j^{\frac{m}{2}-1} \right|,$$

where $\varepsilon_j = \cos\frac{4nj\pi - 2rmk\pi + 4rk\pi}{mn} + i\sin\frac{4nj\pi - 2rmk\pi + 4rk\pi}{mn}$.

Let $b^{\frac{m}{2}}|\lambda E_m - f(\omega_k)| = 0$, then we have

$$\prod_{j=0}^{\frac{m}{2}-1} \left| A' + B'\varepsilon_j + C'\varepsilon_j^{\frac{m}{2}-1} \right| = \prod_{j=0}^{\frac{m}{2}-1} \left| a^2 b - 2b - b^2\varepsilon_j - \varepsilon_j^{-1} \right| = 0. \quad (2)$$

By (2), we can obtain

$$a = \pm 2\cos(2\pi j/m + 2\pi rk/mn).$$

Hence, the eigenvalues of $A(G)$ are

$$\begin{aligned} \lambda_{k+j} &= \left(\omega_k + \omega_k^{n-1}\right) + a \\ &= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} + \cos\frac{2(n-1)k\pi}{n} + i\sin\frac{2(n-1)k\pi}{n} \pm 2\cos\left(\frac{2j\pi}{m} + \frac{2rk\pi}{mn}\right) \\ &= 2\cos\frac{2k\pi}{n} \pm 2\cos\left(\frac{2j\pi}{m} + \frac{2rk\pi}{mn}\right), \end{aligned}$$

where $k = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, \frac{m}{2} - 1$. □

Specially, when $r = 0$, Formula (1) can be gotten by Theorem 2.

4 Enumeration of spanning trees of $P_{m,n,r}$

Theorem 3 *If m is even, then the number of spanning tree of the $P_{m,n,r}$ can be expressed by*

$$\frac{2^{mn}}{mn}\prod_{\substack{k=0\\(k,j)\neq(0,0)}}^{n-1}\prod_{j=0}^{\frac{m}{2}-1}\left[4-4\cos\frac{2k\pi}{n}+\cos^2\frac{2k\pi}{n}-\cos^2\left(\frac{2rk\pi}{mn}+\frac{2j\pi}{m}\right)\right].$$

Proof Note that the degree of every vertex of $P_{m,n,r}$ is 4, the Laplacian matrix $L(G) = 4I_{mn} - A(G)$, and the Laplacian eigenvalues are $u_j = 4 - \lambda_j$, where λ_j are the eigenvalues of $A(G)$. Noticing $u_0 = 0$, we have

$$\begin{aligned} t(G) &= \frac{1}{mn}\prod_{\substack{k=0\\(k,j)\neq(0,0)}}^{n-1}\prod_{j=0}^{\frac{m}{2}-1}\left[4-2\cos\frac{2k\pi}{n}-2\cos\left(\frac{2rk\pi}{mn}+\frac{2j\pi}{m}\right)\right]\\ &\quad\times\left[4-2\cos\frac{2k\pi}{n}+2\cos\left(\frac{2rk\pi}{mn}+\frac{2j\pi}{m}\right)\right]\\ &= \frac{2^{mn}}{mn}\prod_{\substack{k=0\\(k,j)\neq(0,0)}}^{n-1}\prod_{j=0}^{\frac{m}{2}-1}\left[4-4\cos\frac{2k\pi}{n}+\cos^2\frac{2k\pi}{n}-\cos^2\left(\frac{2rk\pi}{mn}+\frac{2j\pi}{m}\right)\right]. \end{aligned}$$

By the definition of the asymptotic tree number entropy, we have

$$\begin{aligned} z(G) &= \lim_{\substack{m\to\infty\\n\to\infty}}\frac{\log t(G)}{mn}\\ &= \ln 2+\lim_{\substack{m\to\infty\\n\to\infty}}\sum_{\substack{k=0\\(k,j)\neq(0,0)}}^{n-1}\sum_{j=0}^{\frac{m}{2}-1}\ln\left[4-4\cos\frac{2k\pi}{n}+\cos^2\frac{2k\pi}{n}-\cos^2\left(\frac{2rk\pi}{mn}+\frac{2j\pi}{m}\right)\right]\\ &= \ln 2+\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\pi}\ln(4-4\cos x+\cos^2 x-\cos^2 y)dxdy \approx 1.1662. \end{aligned}$$

That is the asymptotic tree number entropy of $P_{m,n,r}$ is the same as the one of $P_{m,n}$. □

Acknowledgments This work is supported by NSFC (Grant Nos. 11171279, 11226288, 11271226 and 11301251), promotive research fund for excellent young and middle-aged scientists of Shandong province (Grant No. BS2013DX026) and AMEP of Linyi University

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